

Characterizing Multi-planet Systems with Classical Secular Theory

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Abstract Classical secular theory can be a powerful tool to describe the qualitative character of multi-planet systems and offer insight into their histories. For example. Major axes currently locked in alignment may indicate substantial tidal damping; current near-separatrix behavior may suggest an early mutual scattering event; and eigenmodes help assess dynamical coupling. For 55 Cancri, using the orbital solution by Fischer et al. (2008), the current alignment of major axes is fortuitous as these planets are all controlled by different eigenmodes. Also, there are two dynamical groups: the inner three planets and the outer two. With the orbit solution from Dawson and Fabrycky (2010), again the alignment of major axes is fortuitous, but the grouping is different with the innermost planet decoupled from the other inner planets which has implications for the system's tidal evolution. For 61 Virginis, the libration of the outer planets' pericenter longitudes is not due to eccentricity damping. The outer planets are in a dynamical group and libration results from initial conditions and a propensity due to the system's basic architecture (i.e. masses and semi-major axes). Classical secular theory reproduces the behavior of the hypothetical system discussed by Mardling (2007) despite the eccentricities being large. Gliese 581, HIP 14810, μ Arae, HD 69830, and HD 37124 are also discussed.

Keywords Planetary Systems · Planets · Dissipative Forces · Tides · Secular Theory

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1 Introduction

The acceleration of discoveries of extrasolar systems is providing a base of data for inferring the current properties as well as the histories of planetary systems. Understanding planetary systems in general provides the essential context for our own solar system as well. A system's current dynamical state is the result of the initial planet formation, orbital evolution in the nebula, close planet-planet encounters, and subsequent long-term tidal evolution. Thus its orbital characteristics and distributions provide a key to the systems' origin and evolution. An important step is to find ways to characterize systems that shed light on their history.

An essential part of the initial discovery process is determination of the current keplerian orbital elements. However, mutual perturbations cause these elements to change on timescales short compared with the age of a system. Observers have long been well aware of this issue, so numerical integration is typically part of the orbit-fitting process because it helps rule out ranges of masses and orbital elements that might be consistent with observational data, but inconsistent with long-term stability. Mutual interactions are enhanced by orbital resonances, which are thus especially interesting from the point of view of dynamical theory. However, even in systems with no resonances, secular interactions can be important, and the nature of those interactions can be a key to the history and properties of these systems.

Keplerian orbital elements are constants of integration in the two-body problem, but they are not constant when multiple planets interact. In secular interactions, planets exchange angular momentum, so eccentricities and longitudes of pericenter vary periodically. As long as eccentricities are not too large, the analytical so-

lutions of classical secular theory provide constants of integration that represent the system better than the keplerian elements. While in principle numerical integrations can represent the behavior more precisely, secular theory provides a way to characterize, classify, interpret, and compare the dynamical states of various systems. In this paper, we demonstrate how this characterization can provide insight into the origin and evolution of the system.

The character of a system as used here refers to any of a number of different qualitative behaviors revealed by secular theory. The extent to which a system exhibits certain characteristics defines what we mean by its current dynamical state.

One of the evolutionary processes that governs long-term evolution is the effect of tides. In a single-planet system the effect of tides is comparatively uncomplicated (for example, Jackson et al. 2008[20]). The planet's energy and angular momentum change, affecting the semi-major axis and eccentricity values. The inclination is affected as well, but in this paper we consider only systems in which all orbital and rotational axes are (or can reasonably be assumed to be) aligned. In multi-planet systems, secular interactions redistribute angular momentum among the planets, so that the effect of tides is shared. Over the long term, the result is not only a change in keplerian elements, but also a gradual qualitative change in the secular character of a system. Notably, tidal evolution can lead to an alignment of the orientation of the major axes.

Another past process that can leave a signature on the current secular character of a system is planet-planet scattering. Suppose two planets undergo a close encounter, resulting in large eccentricities (including the possible ejection of one from the system as in the case described by Ford and Rasio 2005[15]). Immediately after the event, the other planets may retain the nearly circular orbits acquired during their formation within a nebula. Then, the subsequent secular interactions must result in periodic changes in all the eccentricities, so that at any observational epoch, all the keplerian eccentricities may be substantial. However, because the secular interactions are periodic, the dynamical behavior in such a case is characterized by periodic returns to circular orbits, i.e. near-separatrix behavior when considered in terms of secular theory (Barnes and Greenberg 2006a)[2].

In this paper we consider several specific planetary systems and types of behavior to demonstrate how the classical treatment of secular theory can reveal the signatures of such past events and evolution. The key to the value of this approach is that classical secular theory can be described by sets of straightforward linear dif-

ferential equations, so that the solutions are expressed as sums of eigenmodes. The constants of integration that characterize the system are the magnitudes and phases of these eigenmodes, rather than keplerian elements like the eccentricities and pericenter longitudes. With understanding of these results, there is the potential to infer significant aspects of the past evolution by inspection of the values of these parameters.

For example, in the presence of secular interactions, any process (such as tides) that tends to damp the eccentricity of one or more planets will actually have the effect of damping all of the eigenmodes, each at a different rate. Eventually, when only the longest lived eigenmode remains, the major axes are aligned, a distinctly recognizable orbital behavior that is the hallmark of eccentricity damping. However, as we show below, major axes being locked in alignment does not necessarily mean that only one eigenmode is dominant. Moreover, as we show with the example of 61 Virginis and HIP 14810 in Sect. 4 and 5, the eigenmode that dominates the behavior of the damped planet is not necessarily the one that damps fastest, contrary to some preconceptions. We also show in Sect. 8 that the relative rates of damping of the eigenmodes provide a useful way to interpret the effects of tidal damping derived in a complementary analysis by Mardling (2007)[23].

For another example of the value of the classical secular approach, consider the signature of planet-planet scattering: A periodic return of at least one planet to a circular orbit. This condition is readily identified by inspection of the magnitudes of the eigenmodes for each planet. If a combination can sum to zero, it means that eccentricities must periodically return to zero. Thus the secular solution can reveal which systems may have suffered a planet-planet scattering event, as discussed further in Sect. 3.1.

The degree to which planets share the effects of the same eigenmodes describes how strongly coupled their behavior is, and thus can shed light on (and help quantify) the closeness of packing of the planets, with important implications for the origin and evolution of the system. The ability to quantify the strength of the coupling is also useful for checking the plausibility of observational orbit fits. This point is demonstrated for the case of 55 Cancri (Sect. 3), where we show that an orbital fit with the inner planet held circular is consistent with the mutual gravitational effects among the planets.

These and other examples of the insight that can be derived from classical secular theory are discussed in Sect. 3 to 7. The underlying theory is comprehensively described in various textbooks (e.g. Brouwer and Clemence 1961[10], Murray and Dermott 2000[24]), and

we review it briefly here (Sect. 2) to make the presentation complete, to establish our notation, and to include the effect of eccentricity damping.

2 Secular Behavior

2.1 Classical Theory

Classical secular theory involves averaging the disturbing force at a planet due to each other planet over the planets' orbital periods. Then the remaining disturbing potential is used to calculate the changes in keplerian orbital elements over time. In effect, the gravitational interactions are computed as if each planet's mass is smeared out along its keplerian orbit with the portion of the mass in any segment of this path determined by how much time the planet would spend there in unperturbed keplerian motion. In practice, the disturbing potential is expanded in a fourier series, and those terms are ignored that explicitly contain orbital longitudes in the arguments of the sines or cosines (Brouwer and Clemence 1961[10], Murray and Dermott 2000[24]). Accordingly, this approach would not be valid where any of the planets are in mean-motion resonance.

Here we also assume that the orbits are co-planar and that the eccentricities are small enough that terms in the disturbing function higher than second order in the eccentricities can be ignored. As we discuss in Sect. 8, even if eccentricity values are fairly large, this approach can still give at least qualitatively meaningful results. It is also quite straightforward to include General Relativity, because its effect on orbital precession has the same functional form as terms in the planetary disturbing function.

The variation equations take on a simple linear form if we replace the keplerian elements eccentricity (e) and longitude of pericenter (ϖ) with the elements h and k :

$$k_p = e_p \cos \varpi_p \quad (1)$$

$$h_p = e_p \sin \varpi_p \quad (2)$$

where the subscript p is an integer denoting the planet, in order from nearest to farthest from the star. h_p and k_p are the cartesian components of the eccentricity vector, whose magnitude is e_p and direction is given by ϖ_p . Expressing Lagrange's equations for planetary perturbations in terms of h and k yields the linear differential equations:

$$\dot{k}_p = \sum_{j=1}^N A_{pj} h_j \quad (3)$$

$$\dot{h}_p = \sum_{j=1}^N A_{pj} k_j \quad (4)$$

where N is the number of planets in the system and the matrix A depends on the masses and semi-major axes of the planets and the mass of the star as given, for example, by Murray and Dermott (2000)[24]. Additional precession effects due to General Relativity, oblateness of the primary, etc., can be accommodated by incorporation in A_{11} , A_{22} , etc.

As standard first-order linear differential equations, the solution of Eqns. 3 and 4 is a sum of eigenmodes:

$$k_p = \sum_{m=1}^N E_m V_{mp} \cos(g_m t + \delta_m) = \sum_{m=1}^N e_{mp} \cos(g_m t + \delta_m) \quad (5)$$

$$h_p = \sum_{m=1}^N E_m V_{mp} \sin(g_m t + \delta_m) = \sum_{m=1}^N e_{mp} \sin(g_m t + \delta_m) \quad (6)$$

where V_{mp} are the eigenvectors and g_m are the corresponding eigenfrequencies. Here we let the subscript m denote the eigenmodes in order of decreasing g_m . The eigenvector V_{mp} for a given mode describes how the power is partitioned among the planets, and it is normalized such that for each mode the root mean square is one. Here, the eigenvectors are normalized so that, for a given m , the root mean square of the N planets' components is unity. The mode strengths E_m and phases δ_m are the constants of integration, while the keplerian elements vary with time. The constants of integration are determined by using the known eccentricities and ϖ values (equivalent to the known h_p, k_p values) at any given time (i.e. initial conditions).

Inspection of Eqns. 6 and 5 shows that for each planet the (h_p, k_p) vector (which we call the eccentricity vector or e vector, with length e_p and direction ϖ_p) is the sum of N component vectors in (h, k) space, each corresponding to a particular eigenmode and rotating in (h, k) space at an angular velocity given by g_m (Figure 1). Moreover, for a given eigenmode, the component vector contributed to the planets are all either parallel or anti-parallel, because they all point in the direction $g_m t + \delta_m$ or $g_m t + \delta_m + \pi$. So, if only one of the eigenmodes has non-negligible strength, all the planets' directions of pericenter would be aligned (if the signs of V_{mp} are the same) or anti-aligned (if the signs of V_{mp} are opposite). Equivalently, we can say the planets' major axes (or lines of apsides) are aligned or anti-aligned.

For each mode, the relative magnitudes of the components V_{mp} describe how strongly that mode affects a given planet compared with the other planets. And for a given planet, the relative magnitudes of the product $E_m V_{mp}$ describes which modes dominate that planet. These considerations can be used to determine how strongly coupled the interactions may be among various subsets of the planets in a system, as discussed in specific examples in the following sections.

Another useful point is that, like the matrix A , the eigenvectors and eigenmodes depend only on the masses and semi-major axes of planets in the system, and the mass of the star. Thus some understanding of the character of the system can be developed even if observations are insufficient to determine the eccentricity values precisely (as demonstrated in Sect. 6 below).

2.2 Secular Behavior with Eccentricity Damping

Suppose some process (in addition to the gravitational effects taken into account in the derivation of Eqns. 5 and 6) acts to damp the eccentricity of the inner planet according to

$$\frac{de_1}{dt} = -F e_1 \quad (7)$$

where F is positive and constant. Such eccentricity damping can be one effect of tidal dissipation (e.g. Goldreich and Soter 1966[16], Yoder and Peale 1981[38], Jackson et al. 2008[20]), although here we are not considering other effects of tides such as changes in the semi-major axis. Eccentricity damping can be incorporated into the secular theory by adding a term $-F h_1$ to dh_1/dt and $-F k_1$ to dk_1/dt in Eqns. 3 and 4, and therefore adding the term $-F i$ to A_{11} (where $i = (-1)^{1/2}$). The secular equations are still linear, so solutions are still straightforward (e.g. Chiang and Murray (2002)[11]). However, the eigenvalues will now be complex, which introduces an exponential decrease of each of the mode strengths, E_m .

Even if the damping process acts directly on only one planet, the secular interactions redistribute angular momentum among the planets, so that eventually all the eccentricities die away. However, each eigenmode damps according to its own exponential timescale. Thus, as a damping proceeds a stage is reached where only the longest-lived eigenmode remains. This condition has been called a fixed point solution or, in recognition of the fact that eventually even the longest-lived eigenmode damps away, a quasi-fixed point solution (Wu and Goldreich 2002[37], Mardling 2007[23]). As long as the one longest-lived eigenmode remains, this condition is characterized by aligned or anti-aligned major axes. Before this condition is reached, but after the other eigenmodes have become smaller than the long-lived mode, their contribution to the total solution of the behavior of the eccentricity vectors is so small that it can only produce librations around the alignment of major axes.

The various properties of the classical solution for secular behavior reviewed in this section have implications for the interpretation of the characteristics of observed extrasolar systems, as demonstrated by the following examples.

3 55 Cancri

The 55 Cancri system consists of five known planets: a super-Earth and four giant planets (Fischer et al 2008[14], Dawson and Fabrycky 2010[12]). Like the terrestrial planets in our own solar system, the relatively small super-Earth is on an orbit interior to that of the giants. Unlike our solar system, the inner planet is close enough to its host star that tides may play a role in the planet's orbital evolution. Table 1 shows the best fit to the radial velocity data for 55 Cancri from the discovery paper by Fischer et al. (2008)[14]. Here, to simplify notation, we refer to these planets by integer in order of increasing semi-major axis rather than by letters in order of discovery. More recent solutions by Dawson and Fabrycky (2010)[12] take aliasing effects into account, yielding a much smaller orbit for the innermost planet. This shorter period has recently been confirmed by transit observations by Winn et al. (2011)[34] with the Microvariability and Oscillations of STars (MOST) telescope and Demory et al. (2011)[13] with the Spitzer Space Telescope.

Here, we consider three different solutions for the orbits in this system as examples to illustrate the value of classical secular theory for describing the dynamical character of a planetary system and identifying its implications. The three systems we consider below are the best-fit from Fischer et al. (2008)[14] and two of the solutions presented by Dawson and Fabrycky (2010)[12]. In all of these solutions several planets have strikingly similar longitudes of pericenter (ϖ), which can be the consequence of eccentricity damping as discussed in Sect. 2.2. Given that the inner planet is close to its star, one might expect tidal dissipation to have been responsible. However, before accepting such a conclusion we must consider the actual magnitude of each eigenmode and how each eigenmode affects the eccentricities of the planets.

3.1 The First Orbit Solution

In the orbit fit by Fischer et al. (2008)[14] (our Table 1), the super-Earth and three giant planets all lie within 1 AU, while the outermost, and much more massive, planet is at about six times that distance; the fourth planet in the system has a nearly circular orbit; and the major axes of planets 1, 2, and 5 are currently nearly aligned, that is they have nearly equal ϖ values.

The eigenmodes derived from the masses and semi-major axis values according to the theory in Sect. 2.1 are shown in Table 2, along with the relative eccentricity damping rates based on the theory in Sect. 2.2. The

actual magnitude (E_m) and phase (δ_m) of each eigenmode is computed from the current e and ϖ values. From those results, Figure 2 shows the current eccentricity vectors for planets 1 through 5, each as a vector sum of its 5 eigenmode components. Remember, for each eigenmode, the corresponding vector component circulates at the rate g_m given in Table 2.

The alignment of the current e vectors for planets 1, 2, and 5 is evident in Figure 2. However, we note that planet 1's e vector is dominated by the 1st eigenmode and planet 5's e vector is dominated by the 5th eigenmode. For planet 2, the e vector is predominantly composed of components from eigenmodes 2 and 3. Because the vector components for each eigenmode rotate at a different rate (g_m) from the other eigenmodes, the current alignment of these three orbits appears to be a coincidence (if not an observational artifact).

The alignment is not the result of damping to a quasi-fixed point. As discussed in Sect. 2.2, if the alignment had been due to eccentricity damping, all but the longest-lived eigenmode would have already damped out. Clearly this is not the case. In fact, the mode that should damp fastest (mode 1 as shown in Table 2) is actually the strongest in the sense of having the greatest amplitude E_m in this system. These arguments show how secular theory can help determine whether or not a system may have experienced tidal evolution.

Another feature of this system is also revealed by consideration of how strongly each planet is affected by each eigenmode. From Table 2 and Figure 2, one can see which planets are controlled by which modes. Specifically: planet 1 is most strongly affected by mode 1, with some contributions by modes 2 and 3; planet 2 is controlled by modes 2 and 3; planet 3 is dominated by mode 2, with an appreciable contribution from mode 3; planet 4 is controlled by modes 4 and 5, with a small contribution from mode 3; and planet 5 is strongly dominated by mode 5. Thus, the eigenmodes that affect the inner three planets do not strongly affect the outer two planets and *vice versa*.

In other words, there are two distinct dynamical groups in the system. In physical terms, the inner three planets exchange angular momentum among themselves, but do not share very much with the outer two planets. Similarly the outer two planets exchange angular momentum predominantly between themselves. In general, the grouping of planets into subsets that share dominant eigenmodes is diagnostic of groups that exchange angular momentum among themselves due to mutual perturbations. An extreme case would be a system with negligible exchanges of angular momentum, in which case each planet would be controlled by only

one eigenmode, and each eigenmode is associated with the behavior of only that one planet.

The dynamical grouping in this system is not what one might have anticipated based on the planets' semi-major axes alone. Because the orbit of planet 5 is much larger than that of any of the others, one might have thought that the inner four planets would be grouped together and the outermost planet would be on its own. But secular theory shows that the dynamical grouping is quite different. This result demonstrates the value of secular theory for helping to characterize the degree of dynamical packing of any system (as long as there are no mean-motion resonances involved).

Additional insight into the two dynamical groups in this system can be gained by further considering the structure of the eigenvectors. For the inner three planets, we see from Table 2 that the second and third eigenmodes have significant values for all of the inner three planets, corresponding to the strong dynamical grouping. However, the first mode is quite concentrated in planet 1. Thus mode 1 provides an additional contribution to planet 1's eccentricity vector, but the link to the other members of this group is only through modes 2 and 3, reflecting the fact that planet 1 is too small to affect the other planets significantly. Accordingly, the damping rate for mode 1 is $0.996F$ (from Table 2) nearly as fast as the direct damping F (as defined in Eqn. 7) of the eccentricity of planet 1, while the other modes damp far more slowly.

In the outer group, the coupling is primarily through the shared eigenmode 5, which is predominantly associated with the outer planet. Mode 5 reflects the effect of planet 5 on planet 4. Mode 4 is essentially only associated with planet 4; the two outer planets are not significantly coupled in this mode, reflecting the small mass of planet 4 and its lack of influence on Planet 5.

One other characteristic of the secular behavior (assuming the small value of e_4 in this orbit fit was correct) is that the initial conditions of the system (which set up the E_m values) were apparently such that planet 4 periodically returns to a circular orbit. Thus the behavior of the outer two planets is similar to the near separatrix secular behavior discussed by Barnes and Greenberg (2006a)[2] for two-planet systems. One way to explain this type of behavior is for planet-planet scattering to have occurred early in the system's history (e.g. Ford and Rasio 2005[15], Malhotra 2002[22], Barnes and Greenberg 2007[5]), an event that set the the initial conditions for the subsequent secular interactions. If planet 4 was not affected by such interactions and remained on an initially circular orbit, while planet 5 was stirred up by a scattering event, then the periodic secular behavior would have to be such that planet 4 periodically

returns to a circular orbit. Hence, this type of on-going secular behavior could be the signature of a long-ago epoch of planet-planet scattering.

3.2 Second Orbit Solution: Best Fit with Aliasing Corrections

Dawson and Fabrycky (2010)[12] reconsidered the orbit solution based on data from Fischer et al. (2008)[14] and more recent observations, taking aliasing effects into consideration. Their best-fit orbital solution (our Table 3), differs from that of Fischer et al. in several ways: the innermost planet is on an orbit much closer to its star; e_4 is much larger; and four of the planets' major axes are currently nearly aligned. Our secular analysis yields the eigenmodes for this solution as shown in Table 4. Figure 3 shows each of the planets' e vectors as a vector sum of the contributions from the five eigenmodes (c.f. Figure 2).

In this system, planets 1, 2, 4, and 5 are all controlled by different eigenmodes. So the current alignment of major axes is fortuitous, as it was for the Fischer et al. (2008)[14] orbits. Also, the eigenmode whose eigenvector is most closely associated with the inner planet (here called mode 2) should damp fastest, yet it has a large E_m , indicating a lack of eccentricity damping (just as for the solution by Fischer et al. 2008[14]).

Inspection of the eigenvectors and their amplitudes reveals similar dynamical groups in Dawson and Fabrycky's best fit to those found for Fischer et al.'s solution. However there are some key differences. First, planet 1 is not nearly as strongly coupled to the other inner planets. In the eigenvectors V_{mp} derived from the Fischer et al. orbits the modes that dominated planets 2 and 3 (modes 2 and 3 in Table 1) also had significant effects on the innermost planet. In contrast, the equivalent modes derived from Dawson and Fabrycky's best fit (modes 1 and 3 in Table 3) barely affect planet 1. This difference reflects the innermost planet's smaller semi-major axis and a consequent relative decoupling from the other planets. (Note, too, that given our convention for ordering the modes, the smaller semi-major axis caused a change in the numbering.)

Another difference is that the 4th eigenmode plays a much greater role in the inner planets' eccentricities in the solution by Dawson and Fabrycky, despite the fact that the normalized eigenvector V_{4p} is nearly the same in both solutions. It has a much greater magnitude (E_4 is ~ 150 times larger) following from the larger value of e_4 and thus both planets 2 and 3 receive noticeable contributions from the 4th eigenmode.

This result demonstrates that two factors influence what constitutes a dynamical grouping of planets: (1)

the distribution of V_{mp} values for one or more modes affecting those planets and (2) the magnitude of the eigenmodes, as both V_{mp} and E_m affect the degree to which various planets are exchanging angular momentum.

3.3 Third Orbit Solution: e_1 Set to Zero

Dawson and Fabrycky (2010)[12] also derived an orbital fit for 55 Cancri in which planet 1's eccentricity was held to zero (Table 5). This constraint was motivated by the idea that the eccentricity should have been damped by tides, given the proximity to the star and reasonable assumptions about tidal parameters. Even with this imposed constraint, this fit was nearly as good as their best fit, according to the chi-squared criterion. However, Dawson and Fabrycky expressed skepticism about this solution, realizing that the influence of other planets in the system could pump up planet 1's eccentricity, in which case a low eccentricity would be short lived and therefore improbable.

Analyzing this system with secular theory (Table 6) we find that the eigenvectors and eigenfrequencies are very similar to those from Dawson and Fabrycky's best fit (Table 4). The only significant difference is found in the eigenmode amplitudes. Most significantly, the magnitude E_2 of mode 2 is now much less than in the best fit case, because this mode is almost exclusively associated with the inner planet, whose eccentricity is now constrained to zero.

The small amplitude of mode 2 is consistent with the fact that this mode should damp fastest according to the damping rate shown in Table 6, and the likelihood that the system has undergone eccentricity damping. However, we note that while small, the magnitude of the mode 2 component of e_1 ($E_2 V_{21}$) is 0.0048, which is not insignificant. Also, modes 1 and 3 contribute a vector component of e_1 with values 0.0029 and 0.0019 respectively. So, although e_1 was artificially set to zero at the time of the observation, over the secular variations its value periodically will cycle up to ~ 0.01 . Thus Dawson and Fabrycky's solution admits interactions that can increase e_1 just as they expected. However, our secular analysis shows e_1 does remain fairly small. Thus the solution with the current e_1 held near zero is reasonably well justified.

This solution is consistent with moderate eccentricity damping acting on the innermost planet, damping E_2 to a fairly small value. However, a solution consistent with advanced eccentricity damping would have mode 2 damped away entirely (i.e. E_2 near zero). Then with even more damping of the inner planet, modes 1

and 3 would damp away next, albeit with damping rates 10^{-4} times that of mode 2 according to Table 6.

The secular analysis thus gives insight into the validity of imposing constraints on solutions for orbital fits, and allows us to interpret the results in terms of a history of tidal evolution. The e_1 solution by Dawson and Fabrycky is a good fit with the data, allows e_1 to remain small even with secular perturbations, and is consistent with the assumption that the inner planet's eccentricity has been significantly damped. However, a solution consistent with mature eccentricity damping would have the 2nd eigenmode fully damped, which is not the case in this solution.

4 61 Virginis

The 61 Virginis system has three planets: a close-in super-Earth and two Neptune-scale planets (Vogt et al. 2010a[32]). Vogt et al. (2010a)[32] obtained the best-fit solution shown in Table 7. They noted that numerical integration of the system revealed that two of the e vectors librate about alignment. Vogt et al. interpreted this condition as indicative of possible tidal evolution. Indeed, as discussed in Sect. 2.2, damping of a planet's eccentricity can lead to such libration after all eigenmodes except one have largely damped away. Our solution for the eigenmodes is shown in Table 8 and Figure 5. The vector sums for planet 2 and planet 3 are currently 27° apart as reported by Vogt et al., and Figure 5 shows that those two planets are dominated by a single eigenmode (mode 3), consistent with the libration noted by Vogt et al.

However, secular theory shows that this behavior does not imply eccentricity damping, whether due to tides or any other process. Although mode 3 dominates the outer two planets, eigenmodes 1 and 2 both still have significant strength. If the inner planet's e had been damped significantly, our solution (Table 8) shows that the first eigenmode should have damped on a timescale 600 times shorter than the other modes. Yet we find (e.g. Figure 5) that mode 1 is still an important component of e_1 . What is more, our solution shows that mode 2, which should have damped out on a timescale comparable to mode 1, is also very strong, so strong that it dominates the inner planet. Thus, while we cannot rule out some small amount of e damping, there is in fact no evidence for tidal damping from the current secular behavior in this case. Rather the libration of the outer two planets about alignment appears to have been determined by the particular initial conditions that determined the subsequent secular behavior of the system.

In fact, for 61 Virginis there is a modest propensity for alignment due to the system's basic architecture. In other words, given the system's masses and semi-major axis values, random values of the initial eccentricity vectors will tend to favor libration about the alignment of pericenter longitudes observed in this system. This modest tendency can be seen by inspection of the eigenvectors (Table 8). With the absolute values of V_{11} and V_{21} both 0.9 or greater, modes 1 and 2 both affect the inner planet most strongly. For that reason, even though the strength E_2 of the second mode is significant, its effect on the outer planets is modest. However, mode 3 affects both of the outer planets with comparable strength ($V_{32} = 0.6$ and $V_{33}=0.8$). Thus, unless mode 2 has an amplitude 50% greater than mode 3, mode 3 will dominate the behavior of the outer planets, resulting in libration similar to what has been reported.

Vogt et al. (2010a)[32] have noted that if the inner planet is terrestrial ($Q \sim 500$, $\rho \sim 6000 \text{ kg/m}^3$), then the eccentricity damping timescale ($1/F$ in our notation) would be much shorter than the age of the system. Because secular theory indicates that any such damping must have been limited at most, we conclude that the innermost planet is most likely Neptune-like ($Q \sim 30000$, $\rho \sim 1500 \text{ kg/m}^3$), with the expected timescale for eccentricity damping is comparable to the several-billion-year age of the system.

The 61 Virginis case also gives insight into the relationship between the eigenvectors and the relative damping rates of the eigenmodes given in Table 8. Recall that the damping of the eigenmodes results from a process that acts to decrease the eccentricity of the inner planet. One might expect the eigenmode that dominates the behavior of the inner planet to be the mode that damps fastest. Table 8 shows that mode 1 is the fastest damping, consistent with this mode being strongly associated with the inner planet ($V_{11}=0.97$). Yet, Figure 5 indicates that mode 2 dominates the inner planet, because the actual effect of each mode over secular timescales is determined by $E_m V_{mp}$, not the normalized eigenvectors (V_{mp}). Mode 2 does damp nearly as fast as mode 1, consistent with its similar strong link to the inner planet ($V_{21}=0.90$). On the other hand, damping of mode 3 is much slower, reflecting the minimal role of this mode in the behavior of the inner planet ($V_{31}=0.11$). This discussion illustrates additional ways that inspection of the eigenvectors in the classical secular solution can provide insight into the character of the orbital interactions.

We discuss this system in more detail in Greenberg and Van Laerhoven (2011b)[18].

5 HIP 14810

The HIP 14810 system (Table 9) consists of a super-Jupiter that is extremely close to the star and two roughly Jupiter-scale planets with semi-major axes of ~ 1 AU (Wright et al. 2009[35]). All three planets have similar, moderate eccentricities.

In this system the inner two planets form a dynamical group (Table 10 and Figure 6). They are coupled by two eigenmodes (modes 1 and 2) whose strength is about equally partitioned between the two planets. The outer planet is decoupled, being controlled only by the 3rd eigenmode, which has little effect on the other planets. Given the eigenvectors, libration of the apsides of the two inner planets will occur if either of the 1st or 2nd eigenmodes is slightly stronger than the other (i.e. larger E_m). Thanks to initial conditions, which give $E_1 \approx 4E_2$, the inner two planets are both dominated by the 1st eigenmode, so there is libration. In this case, because V_{11} has the opposite sign of V_{12} , the apsidal lines librate about anti-alignment.

As in 61 Virginis, the inner planet is not dominated by the mode that damps fastest, which would be the 2nd mode according to Table 10, but instead is dominated by the 1st eigenmode because of the relatively large E_1 . The amplitudes are in fact consistent with some degree of eccentricity damping having occurred, as the 2nd eigenmode's amplitude E_2 is several times smaller than the amplitudes of the other eigenmodes, but it is far from damped completely.

6 Gliese 581

The Gliese 581 system was first identified by Bonfils et al. (2005)[9], who found a single Neptune-scale planet. Since then, several other planets have been discovered, including some super-Earths (Udry et al. 2007[31], Vogt et al. (2010b)[33]). Of the 6 planets inferred by Vogt et al. (2010b)[33], 5 are super-Earths and the other is a Neptune-scale planet (Table 11). All of the detected planets are within 1 AU of the star. The 4th planet outward (g) has attracted considerable attention because it lies in the star's habitable zone (as conventionally defined by the potential for liquid water on its surface). Vogt et al. considered the observations inadequate to reliably extract the eccentricities of the planets, but because the eigenmodes only depend on the masses and semi-major axes, we can use secular theory to help characterize the system. (The solution of Vogt et al. has recently been called into question by Tuomi (2011)[30] and Gregory (2011)[19], but our purpose here is to demonstrate the value of the classical secular theory

approach, so here we discuss the system as described by Vogt et al. (2010b)[33].)

Consideration of the eigenvectors (Table 12) reveals that one, and only one, eigenmode affects the outermost planet, while the five inner planets form a dynamical group defined by shared eigenmodes: The 3rd eigenmode is well spread among the inner four planets; the 4th eigenmode is shared among all five inner planets; and the 5th eigenmode strongly affects both planets 4 and 5. The eigenvectors show that if any of planets 2 through 5 has a significant eccentricity then at least one (probably several) of the other inner five planets will also have (or periodically cycle to) a significant eccentricity.

The dynamical grouping (with the outer planet uncoupled from the rest) has consequences for the distribution of eccentricity damping among the various modes. Because the 1st eigenmode is associated predominantly with planet 1, this mode damps much faster than any other (Table 12). However, because the longer-lived eigenmodes 2 through 4 also significantly couple planet 1 to the others, its eccentricity is also long-lived. It is also worth noting that, because the other inner planets are also quite close to the star, eccentricity damping may act on these planets as well. If such eccentricity damping were to act planet 2 or 3 its effect would be well spread among modes 2 through 4, and thus among all of the five coupled planets.

Subsequent to the analysis of Vogt et al., Anglada-Eschude and Dawson (2011)[1] showed that in fitting the radial velocity data there is a correlation between the eccentricity of planet 5 (d) and the putative existence of planet 4 (g). That study suggests that if planet 4 exists then planet 5 is likely to have an appreciable eccentricity (~ 0.1). In that case, according to the eigenvectors in Table 12, the eccentricity of planet 4 will at least periodically reach a similarly large value. Although this planet nominally lies in the habitable zone, the variation in insolation over its orbit due to this eccentricity would be considerable and could thus affect its habitability.

The above discussion shows how classical secular theory can offer insight into the structure of a multi-planet system even in the absence of knowledge about the planets' eccentricities.

7 Other Systems of Interest

7.1 μ Arae

The μ Arae system consists of four planets (Table 13): a Uranus-scale planet, and three Jupiter-scale planets (Pepe et al. 2007[25]). The secular solution (Table 14

and Figure 7) reveals that, unsurprising given the semi-major axes, the inner and outer planets are decoupled, while the middle two are interacting in a dynamical group.

As can be seen in Figure 7, two of the eigenmodes (1 and 2) are currently at nearly the same phase, and the phases of the other two modes are not very different. In terms of keplerian elements, the omega-tilde values are roughly aligned or anti-aligned. Nevertheless, all the modes have comparable strengths, and the planets in question are controlled by different modes so there is no indication of eccentricity damping. This is another example (like in 55 Cancri) where the current near (anti-)alignment of several orbits is fortuitous.

7.2 HD 69830

The HD 69830 system (Table 15) is composed of three Uranus-scale planets (Lovis et al. 2006[21]). The inner two planets are in a dynamical group and the third planet is decoupled (Table 16 and Figure 8), not surprising given the much greater semi-major axis of planet 3. Again, the amplitudes of the modes are all comparable, showing no sign of eccentricity damping.

7.3 HD 37124

The HD 37124 system (Table 17) has three planets of very similar mass (Wright et al. 2011[36]). The inner planet currently has an eccentricity much smaller than that of others. The secular solution (Table 18 and Figure 9) reveals that the 2nd and 3rd eigenmodes have appreciable power in all three planets, so all three planets are in a single dynamical group. Indeed, off all the V_{mp} , only V_{11} is small (and even it is not so small as to be completely negligible). The current smaller eccentricity of the inner planet (if correct) is only temporary; Its eccentricity is currently near a low point in its secular cycle, as can be seen in Figure 9. The amplitudes of the modes are all comparable, so there is no indication of eccentricity damping.

8 Perspective on an Alternative Formation

Next we consider an artificial system that was introduced by Mardling (2007)[23] to demonstrate an innovative formulation for the evolution of secular interactions. Mardling’s theory was developed to apply even where orbital eccentricities are large, which is a domain where classical secular theory does not apply. However, the trade off is that it only applies in a case of two

planets and only if the ratio of semi-major axes (inner/outer) is small. Thus it is complementary to classical theory, being more accurate for a different range of cases. Numerical integration of Mardling’s equations with tidal evolution showed a damping of the eccentricities, eventually leading to a quasi-fixed state with the pericenter longitudes of the two planets aligned.

The hypothetical system examined by Mardling consisted of two planets orbiting a star of 1.149 solar masses, with semi-major axes of 0.045 AU and 0.4 AU and masses of $0.64 M_J$ and $0.1 M_J$, respectively. The inner planet’s and star’s parameters were inspired by HD 209458 b, while the outer planet was purely hypothetical. The assigned initial conditions (here using our notation) were $e_1=0.1$ and $e_2=0.4$, with pericenters 180° apart. Adopting a value of 100 for the tidal parameter Q , Mardling found the evolution of the eccentricities proceeds as follows (as shown in Figures 3 and 4 of Mardling 2007).

Initially, e_1 oscillates about 0.1 with an amplitude of ~ 0.25 . Over $\sim 10^5$ yr, the amplitude of the oscillation decreases somewhat (to about 2/3 of its initial amplitude) and the value about which it oscillates decreases to a value of about 0.18 (slightly less than half the initial value). The period of the oscillation is fairly constant at about 1.8×10^4 years. (In the case displayed by Mardling, the damping timescale is extremely short: only a few libration periods.)

During this same period of time, the eccentricity of the outer planet undergoes a similar evolution. It initially oscillates about a value of 0.388, with an amplitude of 0.01 and the same frequency as e_1 . During the 10^5 years, the amplitude of the oscillation drops by about a factor of 4, while the values about which it oscillates decreases by $\sim 1\%$. Also during this time, the longitudes of pericenter of the two planets circulate relative to one another, with ϖ_1 advancing at the greater rate.

At 10^5 years the system undergoes a transition to a state where the longitudes of pericenter librate relative to one another. At the same time, the oscillation of e_1 begins to damp at a much faster rate, decreasing by a factor of 3 over the next 10^5 years, while continuing to oscillate at the same frequency.

By about 3×10^5 years after the libration phase began, the oscillations of both eccentricities have died away and $\varpi_1 - \varpi_2$ is locked to zero. During the libration phase, the values about which e_1 and e_2 oscillate gradually decrease. For e_2 it decreases at an exponential rate similar to that during the earlier circulation phase. For e_1 the rate is much slower than in the circulation stage, now close to the exponential rate for e_2 . This phase in

which the oscillations have damped down is sometimes called the fixed-point or quasi-fixed-point condition.

Classical secular theory reproduces this behavior reasonably well, in fact to a remarkable degree given the large eccentricities involved. Applying Mardling's assumed initial conditions, the parameters of the secular solution are the following, as illustrated in Figure 10. For e_2 , mode 2 contributes a component e_{22} of magnitude 0.395, while mode 1's contribution e_{12} is only about 1% as large. For e_1 , mode 2 contributes a component e_{21} of magnitude 0.008, parallel to e_{22} , while mode 1 contributes the larger component e_{11} of magnitude ~ 0.1 , anti-parallel to e_{12} . As also shown in Figure 10, the mode 1 components rotate in direction at a rate $0.026^\circ/\text{yr}$, relative to the mode 2 components (e_{12} and e_{22}). Mardling started her integration at an instant when e_{11} was directed opposite the mode 2 components in the (h, k) space, i.e. when $\varpi_1 - \varpi_2 = 180^\circ$.

As evident from inspection of Figure 10, in the secular solution, as e_{11} circulates (at rate $0.026^\circ/\text{yr}$) relative to mode 2, the value of e_1 oscillates between 0.1 and 0.116 (because $e_{11}=0.108$ and $e_{21}=0.008$), with a period about 1.5×10^4 yr. Pericenter longitude ϖ_1 circulates relative to ϖ_2 . At the same time, the magnitude of e_2 oscillates with an amplitude of $e_{12}=0.005$, as e_{12} circulates in direction. All this behavior is similar to the initial oscillations in the case integrated by Mardling.

With eccentricity damping included in the classical secular theory (Sect. 2.2), we find that mode 1 damps about 10^3 times faster than mode 2. Thus the circular trajectory of the e_{11} vector in (h, k) space (relative to the mode 2 components) will shrink in radius. This shrinking reduces the amplitude of the oscillation, reproducing the amplitude decrease during the circulation phase in Mardling's analysis. Similarly, as the mode 1 component e_{12} shrinks, the oscillations of e_2 decrease in amplitude, again mimicking the behavior found by Mardling. For both planets, the oscillation in e values is about a value given by the mode 2 component, which gradually decreases (1000 times slower than mode 1).

Inspection of Figure 10 also shows that as e_{11} decreases to the same value as e_{21} , the system transitions from circulation to libration. Now e_1 oscillates about e_{21} with an amplitude e_{11} . Eventually mode 1 dies away. With only mode 2 remaining, the system is in the quasi-fixed state, during which the eccentricities gradually damp away on an exponential timescale 1000 times slower than the damping of mode 1.

The evolutionary behavior described by the classical secular approach is similar (even quantitatively) to the results obtained by Mardling in this hypothetical case, which is remarkable considering the large eccentricities involved. Moreover, the classical approach has certain

benefits that complement both numerical integration and Mardling's formulation. Mardling emphasized the distinctions between the phases of the evolution (libration, circulation, and the long-term (quasi-fixed-point) damping. In contrast, the secular solution emphasizes the continuity of this process, with the transitions being simply milestones along the gradual decrease in the magnitudes of the eigenmodes.

While both descriptions are equally valid, the continuous variation emphasized in our classical secular approach is more consistent with the fact that the libration/circulation separatrix in this case is not a boundary between diverging paths in phase space (i.e. it is not like the separatrix between circulation and oscillation of a pendulum for example). Rather, the actual behavior of these planets just before and just after passing from circulation to libration is only incrementally changed.

This example shows that the classical secular approach is not only accurate in a qualitative sense, reproducing the evolutionary behavior derived by Mardling, but it is also surprisingly accurate quantitatively as well. Even though secular theory was developed assuming small eccentricities, in this case all of the critical parameters (oscillation amplitudes, frequencies, relative damping timescales, etc.) are similar to the values from Mardling's solution. Obviously, for some purposes, classical secular theory has limited quantitative value. However, this example shows its value for characterizing the dynamical condition and evolution of a system, even with fairly large e values.

Moreover, the classical theory has several advantages over Mardling's method: Unlike her formulation, it can be applied to a system of any number of planets; it does not require a_1 much less than a_2 ; it does not require an artificially low Q value to obtain a solution; and it gives results in the form of simple analytic expressions that do not need to be numerically integrated. Thus, while not quantitatively precise enough for some applications, the results of classical theory are accurate enough for many purposes.

Of course, where a precise prediction of orbital interactions is required, direct numerical integrations are generally the best approach. However, given the considerable uncertainty in the orbits of most exoplanet systems, the level of precision of numerical integrations for modeling the evolution of a system may be unnecessary. Classical secular theory may provide an analytic approach more commensurate with the actual level of precision of the observations. Moreover, this approach gives insight and a way of intercomparing systems that may not be as evident from numerical integrations of each individual system.

9 Discussion

As shown in this paper, classical secular theory is a useful tool that yields descriptive parameters for characterizing planetary systems in ways that can elucidate current dynamical interactions as well as illuminate and constrain the history of a given system. The method has its limitations of course. As noted above, numerical integrations can yield far more precise ephemerides, given current orbits. On the other hand, where observations are limited, the precision of secular theory could be adequate if commensurate with uncertainties in the observed orbits. The classical theory considered here is also limited by the assumption that there are no significant mean-motion resonances involved. This constraint will limit its applicability in considerations of long-term evolution, during which any system may well pass through resonant conditions. Another limitation is that classical theory as discussed here ignores all terms in the mutual disturbing functions higher than second order in the eccentricities. This limitation is significant because many extra-solar planets have large eccentricities, some approaching unity. However, classical theory is reasonably accurate where eccentricities are modest, and gives qualitatively useful characterizations (and surprisingly accurate quantitative results) even for fairly large eccentricities as shown for example in Sect. 8.

The examples discussed in this paper illustrate various ways that secular theory can provide insight into the dynamics and histories of planetary systems. In each case, the actual current orbits may prove (with further observations) to be different from those we have adopted from the literature. However, that issue is somewhat tangential to the purpose of this paper, which is to demonstrate the value of the classical approach. For that purpose, even hypothetical systems, like the one from Mardling (2007)[23] discussed in Sect. 8 above or the system considered in an illuminating study by Wu and Goldreich (2002)[37], which proved to be non-existent, provide useful demonstrations. As shown here, classical secular theory can provide insight in several ways.

9.1 Dynamical Groups

In each of the systems discussed here, inspection of the relative role of each eigenmode in the behavior of each planet helps characterize how closely the planets are dynamically linked. At one extreme, if each eigenvector had a significant component for only one planet, the planets would be in effect dynamically isolated, the opposite extreme from a packed planetary system (Barnes

and Quinn 2004[8]; Barnes and Raymond 2004[6]; Raymond and Barnes 2005[27], Raymond et al. 2006[28], Barnes and Greenberg 2006b[3]). The dynamical groups that can be inferred from the sharing of eigenmodes (as demonstrated in the cases here) represent the pathways through which planets can exchange angular momentum. They are an indication of the degree of planet-planet interaction, and hence of the denseness of planetary packing.

Strongly coupled groups imply highly packed systems, and *vice versa*. Where a system divides into separate groups of planets only linked among themselves, there is a dynamical gap with possible implications about the completeness of either planet formation or detection. The fact that most planets considered here are secularly coupled supports the notion that planets tend to form in fairly compact configurations.

Nevertheless, we have seen in several cases that subsets of planets are dynamically isolated from others. One example was the system described by Fischer et al.'s (2008)[14] solution for 55 Cancri which divided into two groups: the inner three planets and the outer two planets. In contrast, the solutions for the same system by Dawson and Fabrycky (2010)[12] had only the inner planet relatively uncoupled from the rest. In the case of HIP 14810, the inner two of the three planets are coupled and the outer planet is dynamically alone.

Interestingly, the fact that a set of planets are dynamically linked through a common eigenmode does not necessarily mean that they exchange angular momentum. To exchange angular momentum, the eccentricities of those planets must involve more than one eigenmode so that the contributions from the modes can periodically add constructively and destructively. For a set of planets whose eccentricities are governed by only one eigenmode (i.e. where the other modes have zero amplitudes), all the eccentricity values would be fixed and there would be no exchange of angular momentum.

9.2 Effects of Eccentricity Damping

If any process acts directly to damp the eccentricity of one planet, secular coupling with the other planets will distribute its effects. Thus the magnitudes of all of the eigenmodes of the system will decrease, indirectly damping the eccentricities of all the planets. How fast each eigenmode damps reflects how concentrated (or not) the eigenvector is in the directly affected planet. The damping process that has gotten the most attention is tidal dissipation, which acts most strongly on the inner planet in most systems. Tides also affect semi-major axes, which in turn can have an effect on the rates

of damping of the various components of each eigenvector, as first shown by Wu and Goldreich (2002)[37] for a special case and more generally by Greenberg and Van Laerhoven (2011)[17]. Here though we have only considered the effect of eccentricity damping on the inner planet, and shown how the modification of secular interactions can result in indirect damping of all the orbits.

As the damping process proceeds, a stage is reached where all but the longest-lived eigenmode has damped away. In that case, the single remaining eigenmode behavior is characterized by an alignment of the major axes of the system, accompanied by slowly decreasing eccentricities, with each eccentricity damping at a rate proportional to its magnitude (although this would not be quite true if there were a direct effect changing a planet's semi-major axis (Wu and Goldreich 2002[37], Greenberg and Van Laerhoven 2011[17])). This condition has been called a fixed-point solution in the literature, or a quasi-fixed-point taking into account the slow decrease in the amplitude of the final eigenmode.

The sequential damping of the eigenmodes of the classical secular theory predicts precisely the type of sequence of transitions that was described by Mardling (2007)[23]: First circulation of the pericenter longitudes relative to one another, transitioning to a stage of libration about alignment (or anti-alignment), and then the long-lived quasi-fixed-point behavior with the major axes locked in alignment. The classical theory describes this behavior well, and is surprisingly accurate even when eccentricities might seem to be outside the expected range of validity of the small-eccentricity approximation.

Although a system that has undergone considerable direct damping of one (or more) planet's eccentricity will generally have an alignment of major axes in the quasi-fixed-point condition, an alignment of major axes does not necessarily imply that damping has taken place. We have seen several counter examples, such as the 61 Virginis case, in which an alignment had been interpreted as an indication of tidal evolution (Vogt et al. 2010a)[32], but secular analysis shows that all of the eigenmodes still retain substantial magnitudes so there is no evidence of significant damping. Similarly, in the HIP 14810 system, there is libration about alignment of major axes, but even the fastest-damping eigenmode still has a substantial amplitude, so again the libration is apparently unrelated to any eccentricity damping process.

Furthermore, current alignment or near alignment of planets' lines of apsides do not indicate that those planets are aligned or librating about alignment over the timescales of secular interactions. Examples of such

temporary alignment are seen in the 55 Cancri system (as described by Dawson and Fabrycky's (2010)[12] solution, Sect. 3.2) and the μ Arae system (Sect. 7.1). In both those cases the planets in question are controlled by different modes and the current near alignment is fortuitous.

Given the number of systems with aligned major axes that prove not to result from eccentricity damping process, it may be worth investigating other possible causes. We have seen that such alignments may simply result from initial conditions, with in some cases (e.g. 61 Virginis) some propensity toward alignment due to the particular semi-major axis values and masses in the system. However, an alternative explanation may be that there is some as yet unidentified tendency of the orbit-fitting process to introduce such apparent alignments as a systematic error.

While the quasi-fixed-point solution is the expected final stage of the damping process, this may not always be the case. There is no reason why the damping rates of the longest lived eigenmodes are necessarily very different. In some systems, two or more eigenmodes may damp out on similar timescales, such that there is no single longest-lived mode. In that case, there would never be a quasi-fixed-state (i.e. aligned axes) condition since these eigenmodes will die out together rather than in sequence. Also, if one or more of the outer planets are only weakly coupled to the inner planet then there may be several modes which would not damp appreciably even on timescales as long as the age of the universe or longer. For example, in the 55 Cancri system the 4th and 5th eigenmodes damp much much slower than the other modes and will both therefore persist long term.

9.3 Variation of Eccentricities due to Secular Interaction

Secular theory can help determine whether an eccentricity that is determined (or assumed) to be small will stay that way over secular timescales. For example, in the Gliese 581 case, the first eigenmode damps on a similar timescale to how the inner planet would have damped if no other planets had been present ($\sim 1/F$). But at least two other modes affect the inner planet almost as strongly, and they endure for ~ 50 times longer. As a result, the inner planet's eccentricity will persist longer than it would have were it alone in a single planet system.

Also in Gliese 581, the subgroup of planets 4 (g) and 5 (d) means that if planet d has a substantial eccentricity ~ 0.1 (as seems necessary if planet g exists (Anglada-Eschude and Dawson 2011[1])) then planet g must frequently get about that large, in fact likely

spending much of its time at even larger values. Thus secular theory can play role in consideration of the habitability of this interesting candidate planet.

As a final example, in Dawson and Fabrycky's (2010)[12] orbit-fit solution for 55 Cancri with e_1 held at zero, secular theory allows us to quantify the periodic increase in e_1 's value. We found that the increase was small, so the assumption of small e_1 was reasonably self-consistent. However, because the circular-orbit constraint was motivated by the expectation that the inner planet would have undergone considerable direct damping, a better approach to such a case in the future will be to find a fit in which the shorter-lived eigenmodes have damped away. Such a solution would require new techniques, perhaps an iterative process, but would provide a more well-founded constraint on an orbital fit to account for the likelihood of past eccentricity damping in a system with such a close-in planet.

9.4 Near-separatrix Behavior

Classical secular theory has also provided a way to identify the signature of planet-planet scattering that likely has occurred early in the history of planetary systems, and which may play a major role in setting up their current architecture (Rasio and Ford 1996[26]). Barnes and Greenberg (2006a)[2] (see Barnes and Quinn 2004[8]) showed that the type of scenario demonstrated by Ford and Rasio (2005)[15] produces systems whose secular behavior lies near a separatrix between libration and circulation. Here we have pointed out one example (the system described in Sect. 3.1), but in that case the near-separatrix behavior follows from one planet having a near-circular orbit, which may be an observational artifact. More generally, such systems are very common (Barnes and Greenberg (2006c[4], 2007[5])), so classical secular theory provides a useful tool for identifying systems that may have been influenced by this crucial process.

10 Conclusion

We conclude that classical secular theory is a powerful tool for interpreting the observed dynamical state of a planetary system. Numerical integration is a much more precise way to simulate the on-going behavior of a system, but each integration represents in a sense an anecdotal case. Analytical theory provides a qualitative characterization and classification that allows interpretation and yields implications for the formation and history of these systems. Classical secular theory thus

provides a valuable complement to other approaches to understanding the dynamics of planetary systems.

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Table 1 The orbits of the planets in 55 Cancri as determined by Fischer et al. 2008[14], their Table 4.

Planet	$M \sin(i) (M_J)$	a (AU)	e	$\varpi(^{\circ})$
1 (e)	0.0241	0.038	0.2637	156.5
2 (b)	0.8358	0.115	0.0159	164.0
3 (c)	0.1691	0.241	0.0530	57.4
4 (f)	0.1444	0.785	0.0002	205.6
5 (d)	3.9231	5.901	0.0633	156.6

Table 2 Values pertaining to the eigenmodes of the 55 Cancri system using the orbits found in Table 1 above, Fischer et al.'s (2008)[14] best fit. The eigenvectors (V_{mp}), eigenfrequencies (g_m), and damping rates are all determined using only the masses and semi-major axes of the planets. The "damping rate" is that due to eccentricity damping and is given in terms of F , the direct damping as defined in Eqn. 7. The amplitudes of the eigenmodes (E_m) are determined using the current eccentricities and longitudes of pericenter (ϖ).

Mode	V_{mp} Planet					$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (e)	2 (b)	3 (c)	4 (f)	5 (d)			
1	0.99996	-0.0082	0.0044	-2×10^{-6}	-9×10^{-11}	1.4	$-0.996F$	0.25
2	-0.17	-0.20	0.96	-0.0031	-2×10^{-7}	0.81	$-0.0016F$	0.053
3	0.34	0.77	0.55	-0.020	-6×10^{-7}	0.15	$-0.0028F$	0.021
4	-0.0038	-0.0093	-0.012	-0.9999	0.00045	0.019	$-5 \times 10^{-7}F$	0.0017
5	0.00013	0.00033	0.00043	0.034	0.9994	0.000062	$-8 \times 10^{-12}F$	0.063

Table 3 The best fit orbits of the planets of 55 Cancri as found by Dawson and Fabrycky (2010)[12], their Table 7.

Planet	$M \sin(i) (M_J)$	a (AU)	e	$\varpi(^{\circ})$
1 (e)	0.0261	0.01564	0.17	177
2 (b)	0.826	0.1148	0.014	146
3 (c)	0.171	0.2403	0.05	95
4 (f)	0.150	0.781	0.25	180
5 (d)	3.83	5.77	0.024	192

Table 4 Values pertaining to the eigenmodes of the 55 Cancri system using the orbits found in Table 3 above, Dawson and Fabrycky's (2010)[12] best fit scenario (c.f. Table 2).

Mode	V_{mp} Planet					$g_m(^{\circ}/yr)$	$\frac{\dot{E}_m}{E_m}$	E_m
	1 (e)	2 (b)	3 (c)	4 (f)	5 (d)			
1	-0.053	0.20	-0.98	0.0033	3×10^{-7}	0.81	$-0.000098F$	0.040
2	-0.999996	0.00069	0.0026	-0.000014	-1×10^{-9}	0.62	$-0.9998F$	0.17
3	0.087	0.82	0.57	-0.024	-6×10^{-7}	0.14	$-0.00011F$	0.022
4	-0.00095	-0.011	-0.013	-0.9998	0.00050	0.019	-2×10^{-8}	0.25
5	0.000033	0.00040	0.00049	0.035	0.9994	0.000068	-4×10^{-13}	0.024

Table 5 An alternative fit with $e_1=0$ for the planets of 55 Cancri as found by Dawson and Fabrycky (2010)[12], their Table 8.

Planet	$M \sin(i) (M_J)$	a (AU)	e	$\varpi(^{\circ})$
1 (e)	0.0258	0.01564	0.0	0
2 (b)	0.825	0.1148	0.012	147
3 (c)	0.172	0.2402	0.06	99
4 (f)	0.150	0.781	0.13	180
5 (d)	3.83	5.77	0.029	189

Table 6 Values pertaining to the eigenmodes of the 55 Cancri system using the orbits found in Table 5 above, Dawson and Fabrycky's (2010)[12] alternative fit with $e_1 = 0$ (c.f. Table 2).

Mode	V_{mp}					$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (e)	2 (b)	Planet 3 (c)	4 (f)	5 (d)			
1	0.060	-0.21	0.98	-0.0033	-3×10^{-7}	0.77	$-0.00013F$	0.049
2	0.999996	-0.00059	-0.0028	0.000014	1×10^{-9}	0.62	$-0.9998F$	0.0048
3	-0.083	-0.82	-0.57	0.024	6×10^{-7}	0.13	$-0.00010F$	0.023
4	0.00091	0.011	0.013	0.9998	-0.00050	0.018	$-2 \times 10^{-8}F$	0.13
5	0.000032	0.00040	0.00049	0.035	0.9994	0.000065	$-4 \times 10^{-13}F$	0.029

Table 7 The orbits of the planets in the 61 Virginis system as determined by Vogt et al. (2010a)[32].

Planet	$M \sin(i) (M_E)$	a (AU)	e	$\varpi(^{\circ})$
1 (b)	5.1	0.0501	0.12	105
2 (c)	18.2	0.2175	0.14	341
3 (d)	22.9	0.476	0.35	314

Table 8 Values pertaining to the eigenmodes of the 61 Virginis system using the masses and semi-major axes determined found in Table 7 above, as determined by Vogt et al. (2010a)[32] (c.f. Table 2).

Mode	V_{mp}			$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (b)	2 (c)	3 (d)			
1	0.97	-0.24	0.081	0.039	$-0.65F$	0.064
2	-0.90	-0.40	0.16	0.035	$-0.35F$	0.24
3	0.11	0.58	0.81	0.0092	$-0.0011F$	0.39

Table 9 The orbits of the planets in the HIP 14810 system as determined by Wright et al. (2009)[35].

Planet	$M \sin(i) (M_J)$	a (AU)	e	$\varpi(^{\circ})$
1 (b)	3.88	0.0692	0.143	159
2 (c)	1.28	0.545	0.143	329
3 (d)	0.570	1.89	0.164	286

Table 10 Values pertaining to the eigenmodes of the HIP 14810 system using the masses and semi-major axes from Table 9 above, as determined by Wright et al. 2009[35] (c.f. Table 2).

Mode	V_{mp}			$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (b)	2 (c)	3 (d)			
1	0.54	-0.83	0.085	0.057	$-0.31F$	0.19
2	-0.82	-0.57	0.085	0.043	$-0.69F$	0.051
3	0.017	0.096	0.995	0.013	$-0.00040F$	0.17

Table 11 The masses and semi-major axes of the six planets in the Gliese 581 system according to Vogt et al. (2010b)[33]. Eccentricity values have not been determined.

Planet	$M \sin(i) (M_E)$	a (AU)
1 (e)	1.7	0.0284
2 (b)	15.6	0.0406
3 (c)	5.6	0.0730
4 (g)	3.1	0.1460
5 (d)	5.6	0.2185
6 (f)	7.0	0.758

Table 12 Values pertaining to the eigenmodes of the Gliese 581 system using the masses and semi-major axes found in Table 11 above, as determined by Vogt et al. 2010[33] (c.f. Table 2). E_m values cannot be computed because the eccentricities and longitudes of pericenter ($varpi$) are unknown.

Mode	V_{mp} Planet						$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$
	1 (e)	2 (b)	3 (c)	4 (g)	5 (d)	6 (f)		
1	-0.997	0.079	-0.0045	-0.000063	-9×10^{-6}	-3×10^{-8}	6.4	$-0.94F$
2	0.35	0.38	-0.86	0.035	0.0016	6×10^{-6}	1.0	$-0.022F$
3	-0.48	-0.56	-0.54	0.40	-0.064	0.000040	0.32	$-0.038F$
4	-0.14	-0.17	-0.19	-0.92	0.26	-0.00019	0.26	$-0.0044F$
5	-0.025	-0.030	-0.048	-0.51	-0.86	0.0076	0.055	$-0.000078F$
6	0.00044	0.00054	0.00092	0.0096	0.018	0.9998	0.0017	$-9 \times 10^{-9}F$

Table 13 The orbits of the planets in the μ Arae system as determined by Pepe et al. (2007)[25].

Planet	$M \sin(i)$ (M_J)	a (AU)	e	$\varpi(^{\circ})$
1 (c)	0.03321	0.09094	0.172	212.7
2 (d)	0.5219	0.9210	0.0666	189.6
3 (b)	1.676	1.497	0.128	22.0
4 (e)	1.814	5.235	0.0985	57.6

Table 14 Values pertaining to the eigenmodes of the μ Arae system using the masses and semi-major axes found in Table 13 above, as determined by Pepe et al. 2007[25] (c.f. Table 2).

Mode	V_{mp} Planet				$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (c)	2 (d)	3 (b)	4 (e)			
1	0.0017	-0.98	0.20	-0.00050	0.31	$-5 \times 10^{-8}F$	0.15
2	0.027	-0.63	-0.77	0.032	0.033	$-5 \times 10^{-6}F$	0.12
3	-0.9999998	-0.00015	-0.00014	0.000020	0.013	$-0.999995F$	0.17
4	0.0056	0.056	0.073	0.996	0.0036	$-8 \times 10^{-8}F$	0.10

Table 15 The orbits of the planets in the HD 69830 system as determined by Lovis et al. (2006)[21].

Planet	$M \sin(i)$ (M_E)	a (AU)	e	$\varpi(^{\circ})$
1 (b)	10.2	0.0785	0.10	340
2 (c)	11.8	0.186	0.13	221
3 (d)	18.1	0.630	0.07	224

Table 16 Values pertaining to the eigenmodes of the HD 69830 system using the masses and semi-major axes found in Table 15 above, as determined by Lovis et al. 2006[21] (c.f. Table 2).

Mode	V_{mp} Planet			$g_m(^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (b)	2 (c)	3 (d)			
1	0.92	-0.40	0.0041	0.069	$-0.74F$	0.14
2	-0.62	-0.79	0.030	0.024	$-0.26F$	0.10
3	0.046	0.087	0.995	0.0024	$-0.00042F$	0.074

Table 17 The orbits of the planets in the HD 37124 system as determined by Wright et al. (2011)[36].

Planet	$M \sin(i)$ (M_J)	a (AU)	e	$\varpi(^{\circ})$
1 (b)	0.675	0.53364	0.054	130
2 (c)	0.652	1.7100	0.125	53
3 (d)	0.696	2.807	0.16	0

Table 18 Values pertaining to the eigenmodes of the HD 37124 system using the masses and semi-major axes found in Table 17 above, as determined by Wright et al. (2011)[36] (c.f. Table 2).

Mode	V_{mp} Planet			$g_m (^{\circ}/yr)$	$\frac{1}{E_m} \frac{dE_m}{dt}$	E_m
	1 (b)	2 (c)	3 (d)			
1	-0.092	0.86	-0.51	0.080	-0.0045 F	0.087
2	0.95	-0.17	-0.28	0.023	-0.80 F	0.14
3	0.59	0.53	0.61	0.014	-0.20 F	0.18

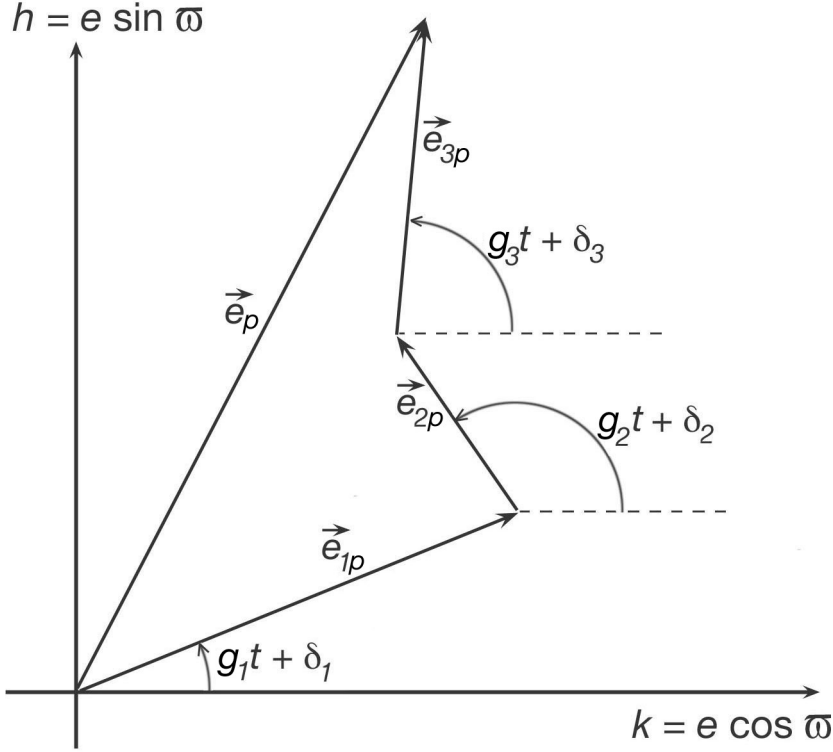


Fig. 1 The behavior of the eccentricity of planet p according to the secular solution (Sect. 2). The contribution of each mode to this planet is $e_{mp} = E_m V_{mp}$, and it rotates at a rate given by the corresponding eigenfrequency, g_m . The phase δ_m and the amplitudes of the modes E_m are set by initial conditions.

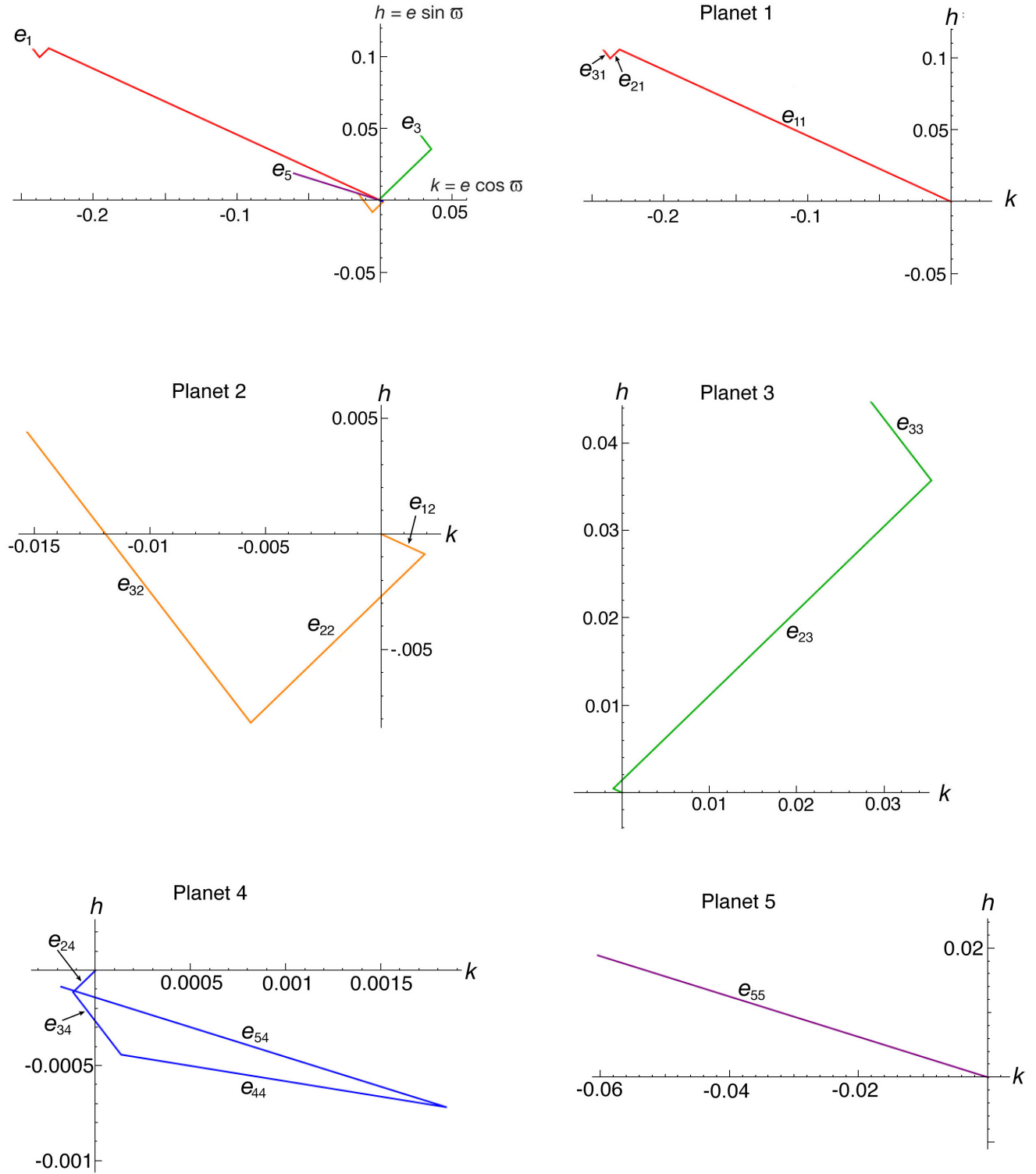


Fig. 2 The eccentricity vectors (k, h) for each of the planets of 55 Cancri shown as a vector sum of the contributions from the five eigenmodes based on the Fischer et al. (2008)[14] orbit solution (Table 1). As plotted here, mode 1 starts from the origin, mode 2 starts from the end of mode 1, etc. The planets are colored from hotter to cooler colors in order of increasing semi-major axis: red, orange, green, blue, and purple for planets 1–5, respectively. Planets 1, 2, and 5 are all dominated by different eigenmodes. Remember that the contributions from a given eigenmode must be parallel or anti-parallel (i.e., a 180 degree difference). Top left: the (k, h) plots for all planets combined for comparison on a common scale. Other panels show the (k, h) plots for individual planets.

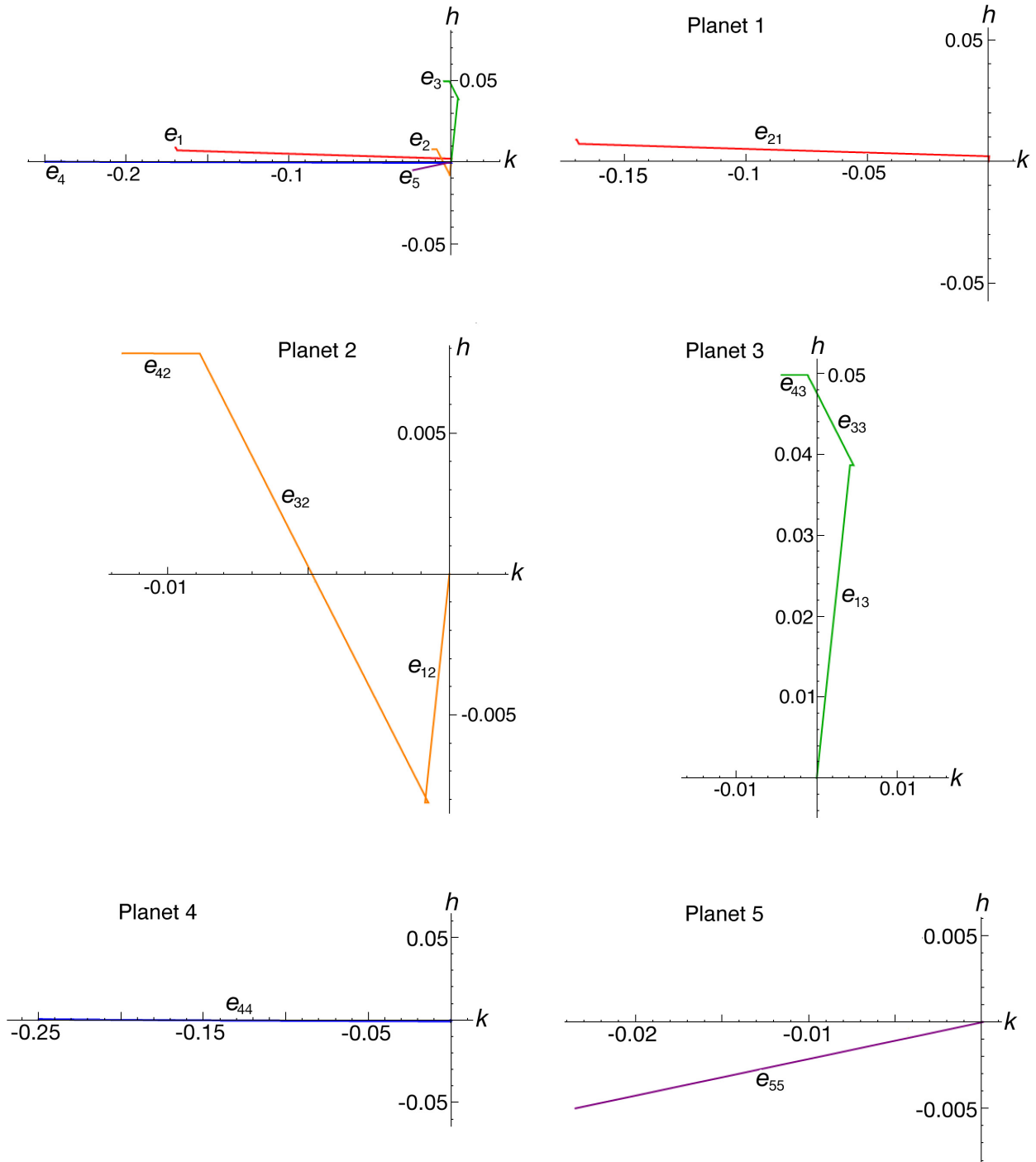


Fig. 3 The eccentricity vectors (k, h) for the planets of 55 Cancri shown as a vector sum of the contributions from the five eigenmodes based on the best fit orbit solution by Dawson and Fabrycky 2010 (Tables 3 and 4). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. Again, it is obvious that planets 1, 2, 4, and 5 are dominated by different eigenmodes.

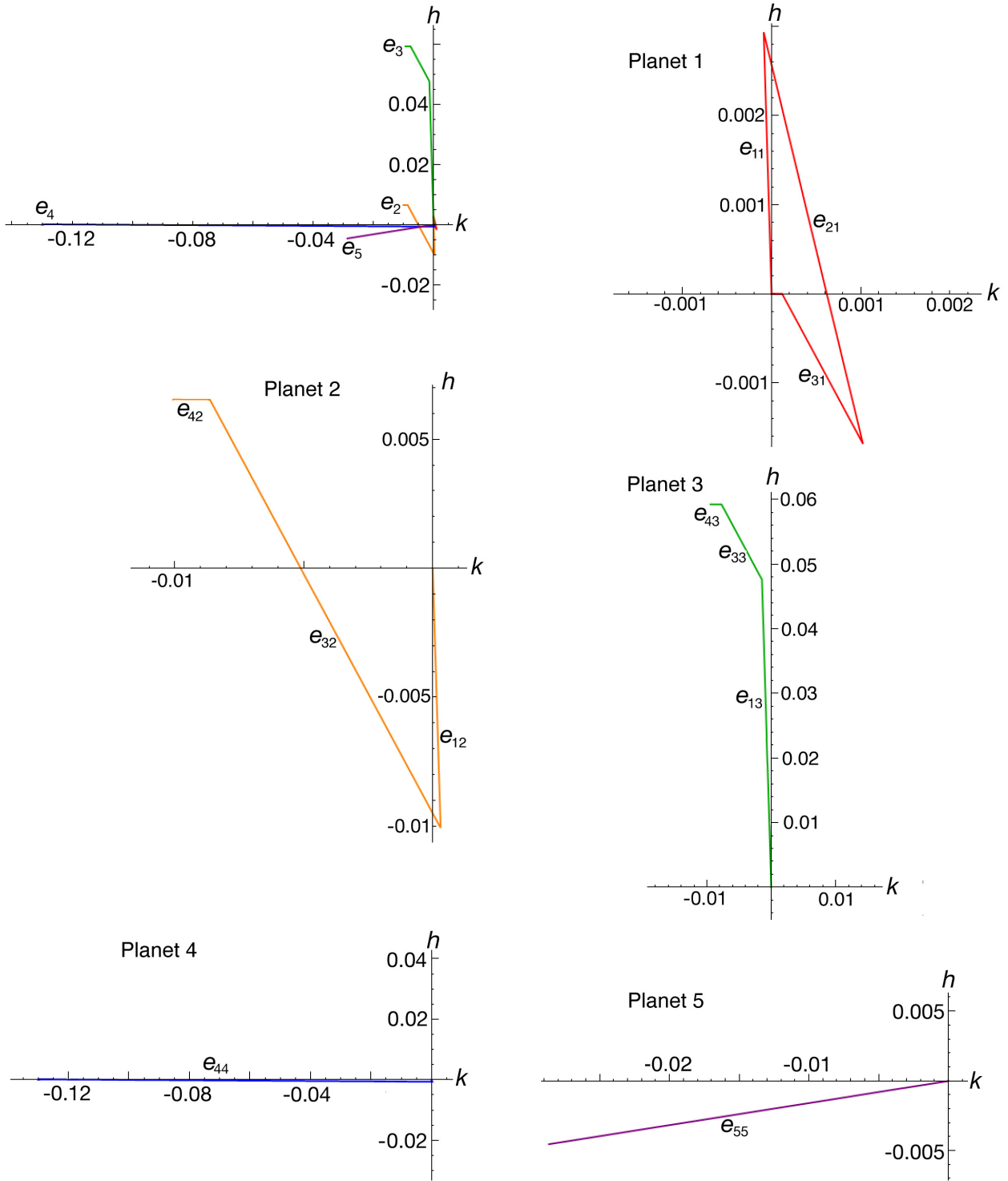


Fig. 4 The eccentricity vectors (k, h) for the planets of 55 Cancri shown as a vector sum of the contributions from the five eigenmodes based on the alternative fit with $e_1 = 0$ by Dawson and Fabrycky 2010 (Tables 5 and 6). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. Planets 2, 4, and 5 are still fortuitously currently near alignment. The overall qualitative character of the system has not changed from Table 4 and Figure 3 with the exception that planet 1 now will not cycle to a very high eccentricity.

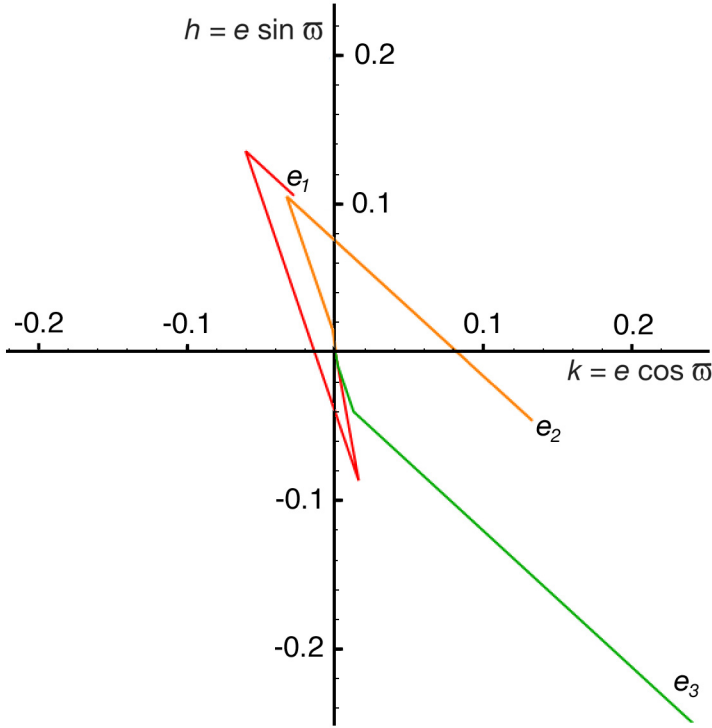


Fig. 5 The eccentricity vectors (k, h) for the planets of 61 Virginis shown as a vector sum of the contributions from the three eigenmodes based on the orbits determined by Vogt et al. (2010)[32] (Tables 7 and 8). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. It is clear here that all modes have appreciable amplitudes and that the 3rd eigenmode strongly dominates the behavior of the outer two planets.

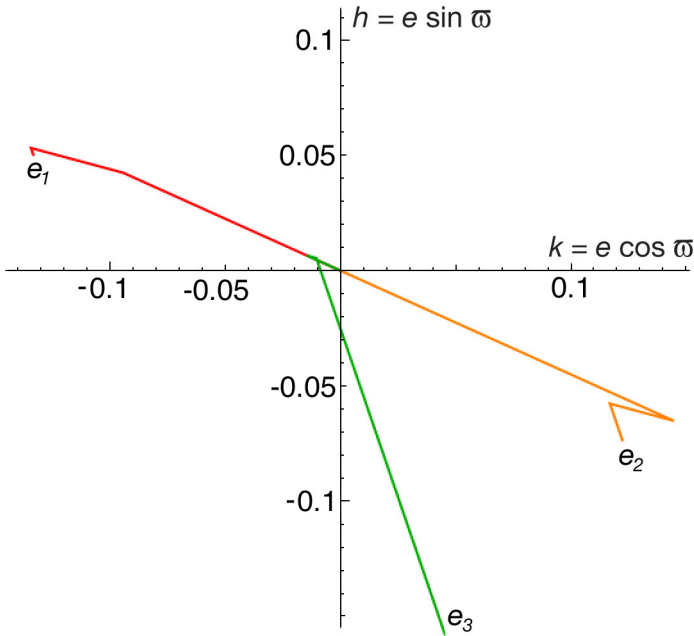


Fig. 6 The eccentricity vectors (k, h) for the planets of HIP 14810 shown as a vector sum of the contributions from the three eigenmodes determined using the orbits determined by Wright et al. (2009)[35] (Tables 9 and 10). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. Here it can be seen that the 1st and 2nd eigenmodes significantly affect the inner two planets (red and orange), but do not strongly effect planet 3 (green). The smaller amplitude of the 2nd eigenmode is also apparent.

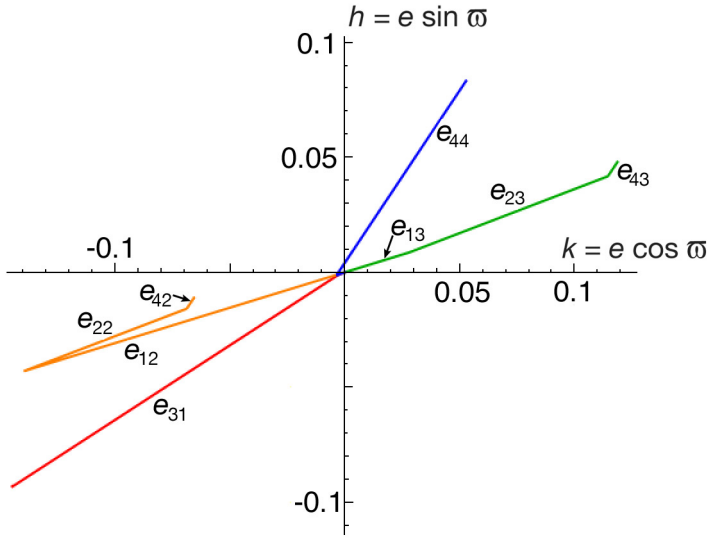


Fig. 7 The eccentricity vectors (k, h) for the planets of μ Arae shown as a vector sum of the contributions from the four eigenmodes determined using the orbits determined by Pepe et al. (2007)[25] (Tables 13 and 14). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. Here it is obvious that the inner planet (red) and outer planet (blue) are uncoupled from the rest of the system. They are each completely controlled by a single (different) mode, neither of which strongly influence any other planet. The middle planets (orange and green) interact via modes 1 and 2.

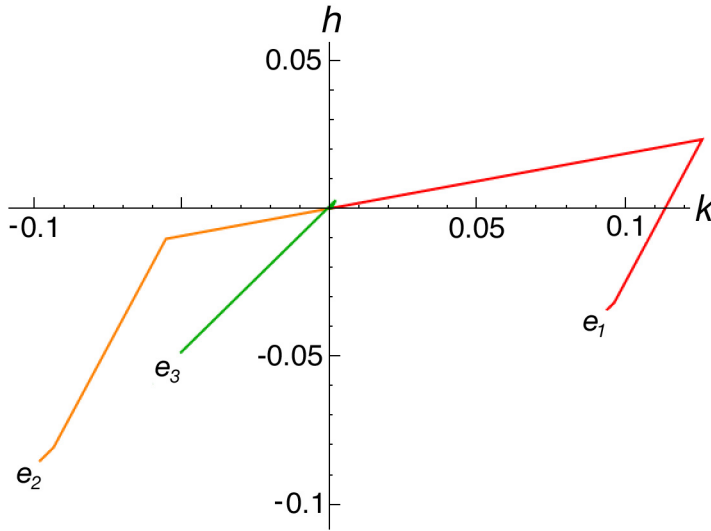


Fig. 8 The eccentricity vectors (k, h) for the planets of HD 69830 shown as a vector sum of the contributions from the three eigenmodes determined using the orbits determined by Lovis et al. (2006)[21] (Tables 15 and 16). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. The outer planet (green) is independent, while the inner two planets interact via modes 1 and 3.

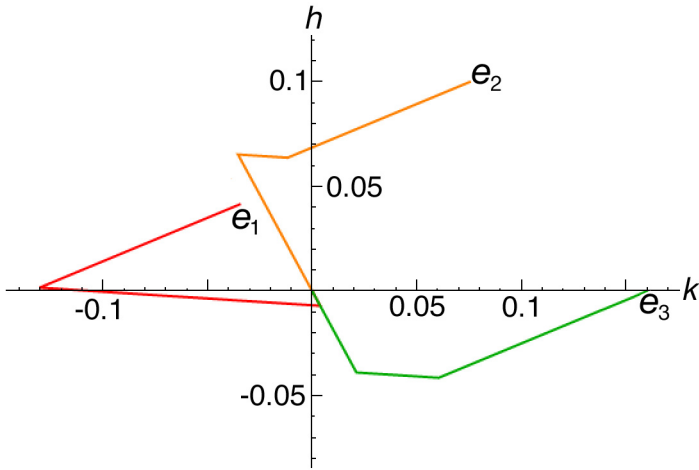


Fig. 9 The eccentricity vectors (k, h) for the planets of HD 69830 shown as a vector sum of the contributions from the three eigenmodes determined using the orbits determined by Wright et al. (2011)[36] (Tables 17 and 18). The planets are colored from hotter to cooler colors in order of increasing semi-major axis. All three planets are interacting.

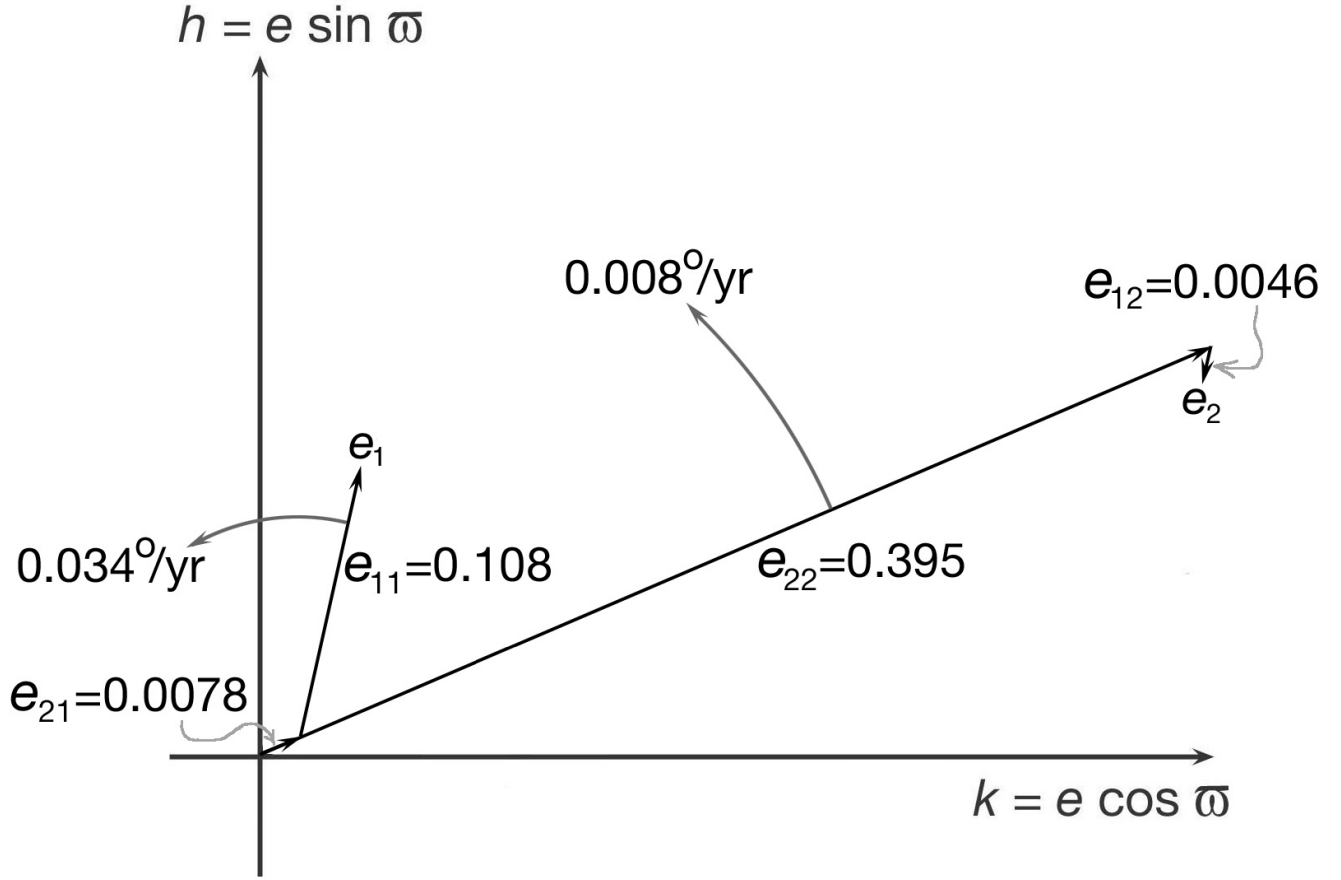


Fig. 10 With the parameters and initial conditions of the hypothetical system investigated by Mardling (2007)[23], classical secular theory yields the behavior of the system illustrated here. The eccentricity e_1 of the inner planet is given by the vector sum in (h, k) space of components e_{11} from eigenmode 1 and e_{21} from eigenmode 2, with values shown. Similarly, the eccentricity e_2 of the outer planet is the sum of components e_{12} (always directed opposite e_{11}) and e_{22} (aligned with e_{11}). As e_{11} circulates at the rate $0.026^\circ/\text{yr}$ relative to the mode 2 components, $\varpi_1 - \varpi_2$ circulates through 360° . Later as mode 1 damps down (so so $e_{11} \downarrow e_{21}$), $\varpi_1 - \varpi_2$ librates. The behavior and evolution closely follows that derived by Mardling.